

# Relative complements and a ‘switch’-classification of simple graphs

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## Abstract

In the paper we introduce and study a classification of finite (simple, undirected, loopless) graphs with respect to a switch-equivalence (‘local-complement’ equivalence of [4], an analogue of the complement-equivalence of [3]). In the paper we propose a simple inductive method to compute the number of switch-types of graphs on  $n$  vertices and we show that there are exactly 16 such types of graphs on 6 vertices.

*keywords:* complete graph, bipartite graph, local complement (in a point).

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## Introduction

In the paper we introduce and study a classification of (simple, undirected, loopless) graphs which was defined, in fact, many years ago as a convenient tool to classify configurations of some sort (see [4], [5]). Namely such a graph is used as a parameter of construction of a so called multiveblen configuration. And the resulting configurations are isomorphic when corresponding graphs are equivalent exactly in the sense considered in this paper. However, at its origins, the equivalence in question was just an auxiliary notion. Its definition was rather (formally) complicated, though it was easy to use when one wanted to check ‘by hand’ whether two given graphs are equivalent. This original definition, now considered rather as a criterion, turned out to be equivalent to a very elegant one, which can be briefly presented as follows. It is a folklore that the binary symmetric difference operation determines the structure of an abelian group on the family  $\wp(\mathcal{X})$  of all subsets of a fixed set  $\mathcal{X}$ , for every  $\mathcal{X}$ . In particular, we can take  $\mathcal{X}$  to be the set of edges of a complete graph  $K_X$ . The set of all complete bipartite subgraphs of  $K_X$  yields a subgroup, isomorphic to  $2^{|X|-1}$ . And, finally, two graphs defined on  $X$  are equivalent when they are congruent modulo the group of complete bipartite subgraphs defined on  $X$ .

In our opinion, there are two arguments which prove that this equivalence is worth studying. The first argument follows from the way in which it is defined: it places our notion within investigations on natural elementary algebra of sets. The second argument consists in interesting interpretation in terms of ‘switches’. Imagine that a graph  $\mathcal{G}$  characterizes possible connections between  $n$  places. Another graph

$\mathcal{G}'$  is equivalent to  $\mathcal{G}$  if it arises from  $\mathcal{G}$  when in some places all existing connections are blocked, and all the other, previously blocked, become unblocked.

This idea has appeared in the mathematical literature (especially with connections to computer sciences) many years ago, and there is a great amount of papers where complement-equivalence, switching-equivalence and related notions were introduced and studied in the class of *directed* graphs (digraphs). Just to quote some of them: [3], [1], [2].

Our equivalence has  $2^{\binom{|X|}{2}-|X|+1}$  equivalence classes and this number rapidly increases when  $|X|$  increases.

Quite often we are interested not in a concrete graph (a realization) but more in its isomorphism type. The number  $\mu_n$  of isomorphism types of respective congruence classes grows up not so rapidly, at least for small values of  $n = |X|$ . But the exact formula for  $\mu_n = \mu(n)$  is hard to find. The reasoning used to compute the size of a congruence class cannot be applied now, as in a congruence class various numbers of pairwise nonisomorphic graphs may appear. For example, for  $n = 4$ , there are three iso-types of graphs equivalent to  $K_4$  and five iso-types of graphs equivalent to  $L_4$ .

In the paper we propose an inductive method to compute  $\mu_n$ . It is evident that  $\mu_3 = 2$  and it was proved in [4] that  $\mu_4 = 3$  and  $\mu_5 = 7$ . Here we show how our machinery gives  $\mu_6 = 16$ . This is one, particular result of the paper. At the same time we determine fundamental general properties of the equivalence introduced, and show several general invariants of this equivalence.

Finally, it is worth to note that the family of all complete bipartite graphs defined on  $X$  together with all disjoint unions of pairs of complete subgraphs of  $K_X$  is also a subgroup of all subgraphs of  $K_X$ . Consequently, congruence modulo this subgroup also defines an equivalence of graphs. One can note that two graphs are equivalent in this sense if either they are equivalent in the sense introduced in the paper or one is equivalent to the boolean completion of the second. An interpretation of this equivalence in terms of switches is also possible, but now one should pay attention more to a binary labeling connected/unconnected, in fact: a labelling of the edges of  $K_X$  by two distinct symbols.

## 1 Basic definitions and facts

### 1.1 Graph-theoretical notations

The equivalence of graphs investigated in the paper is closely related to a classification of partial Steiner triple systems of some sort. However, the resulting classification of graphs has its own interest; it has quite natural intuitions and motivations concerning flows on (undirected) graphs.

Let  $X$  be a nonempty set; then  $\wp(X)$  is the set of all subsets of  $X$ . For an integer  $k$  we write  $\wp_k(X)$  for the set of  $k$ -subsets of  $X$ . A *graph* (an undirected graph without multiplied edges and loops defined on a set  $X$  is an arbitrary subset  $\mathcal{E}$  of  $\wp_2(X)$ ; if  $\{x, y\} = e \in \mathcal{E}$  we say that  $x, y$  are the *vertices* of the *edge*  $e$ . More precisely, we sometimes say that a graph is the structure  $\langle X, \mathcal{E} \rangle$ : if  $\mathcal{E} \subset \wp_2(Y)$  for  $Y \subsetneq X$  this caution is necessary. Clearly, if  $G = \langle X, \mathcal{E} \rangle$  is a graph then  $\varkappa(G) = \langle X, \varkappa(\mathcal{E}) \rangle$  with

$\varkappa(\mathcal{E}) = \wp_2(X) \setminus \mathcal{E}$  is also a graph defined on  $X$ . Most of the notions concerning graphs used in the paper are standard and can be found in any standard textbook, like e.g [6]

Recall the definition of the symmetric difference operation  $\div$  defined on the family of sets:  $A \div B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$ . Recall also that  $\div$  defines on each set  $\wp(W)$ , with  $W$  arbitrary, the structure of an abelian group with  $\emptyset$  as the unit and each element of order 2.

Two operations on graphs will be frequently used in the paper: Let  $G_i = \langle X_i, \mathcal{E}_i \rangle$  for  $i = 1, 2$ .

$$G_1 \div G_2 := \langle X, \mathcal{E}_1 \div \mathcal{E}_2 \rangle \quad (1)$$

$$\text{when } X_1 = X_2 = X,$$

$$G_1 \cup G_2 := \langle X_1 \cup X_2, \mathcal{E}_1 \cup \mathcal{E}_2 \rangle \quad (2)$$

$$\text{when } X_1 \cap X_2 = \emptyset.$$

Several types of graphs are frequently considered in the literature; these will also play a crucial role in forthcoming classifications.

$K_X = \langle X, \wp_2(X) \rangle$ : the *complete* graph on  $X$ ;

$N_X = \langle X, \emptyset \rangle = \varkappa(K_X)$ : the *empty* graph (note that in this case ‘a graph’ must be considered as ‘a structure’);

$K_n, N_n$  : an arbitrary graph isomorphic to  $K_X, N_X$  with  $|X| = n$ ; clearly, the isomorphism type of a complete graph and of an empty graph depends only on the cardinality of its set of vertices.

$C_n$  : a *cyclic* graph on a  $n$ -set;

$L_n$  : a *linear* graph on  $n$  vertices;

$K_{A,B} = \langle A \cup B, \{\{a, b\} : a \in A, b \in B\} \rangle$  defined when  $A \cap B = \emptyset$ ,  $A \neq \emptyset$  or  $B \neq \emptyset$ , a *complete bipartite* graph on  $A \cup B$ .

$K_{n_1, n_2}$  : a graph  $K_{A,B}$  with  $|A| = n_1, |B| = n_2$ .

Finally, we set

$$\mathcal{G}(X) := \{ \langle X, \mathcal{E} \rangle : \mathcal{E} \subset \wp_2(X) \}, \quad (3)$$

$$\mathcal{D}(X) := \{ K_{A, X \setminus A} : A \subset X \}. \quad (4)$$

In most parts of the paper we shall omit proofs of elementary set theoretic formulas, like e.g. (7).

## 1.2 Equivalences of graphs

Let us introduce the following two relations  $\cong$  and  $\approx$  defined on the family  $\mathcal{G}(X)$ . Let  $G_1, G_2 \in \mathcal{G}(X)$ .

$$G_1 \cong G_2 \iff G_1, G_2 \text{ are isomorphic}, \quad (5)$$

$$G_1 \approx G_2 \iff G_1 \div G_2 \in \mathcal{D}(X). \quad (6)$$

The following formula is valid for any  $A, B \subset X$ :

$$K_{A, X \setminus A} \div K_{B, X \setminus B} = K_{A \div B, X \setminus (A \div B)}. \quad (7)$$

As an immediate consequence of (7) and properties of the operation  $\div$  we infer

**PROPOSITION 1.1.** *The class  $\mathcal{D}(X)$  is closed under  $\div$ . Consequently, the relation  $\approx$  is an equivalence in  $\mathcal{G}(X)$ .*

Next, we note another set theoretical relation:

$$K_X \div K_{A, X \setminus A} = K_A \cup K_B. \quad (8)$$

These two: (8) and (7) give us immediately two equivalence classes of  $\approx$ :

$$[K_X]_{\approx} = \{K_A \cup K_B : A \cup B = X, A \cap B = \emptyset\}, [N_X]_{\approx} = \mathcal{D}(X). \quad (9)$$

One more formula of this type can be also worth to note:

$$(K_A \cup N_{X \setminus A}) \div K_{A, X \setminus A} = K_X \setminus K_{X \setminus A} = \varkappa(K_{X \setminus A} \cup N_A), \quad (10)$$

so in one  $\approx$ -class are: a complete subgraph and the boolean complement of suitable another complete subgraph.

In the original paper [4] the definition of  $\approx$  was introduced with the help of the operation of *local complementation*: for  $a \in X$  and  $\mathcal{E} \in \mathcal{G}(X)$  we proceed as follows.

Let  $e \in \wp_2(X)$ . If  $a \notin e$  then  $e \in \mu_a(\mathcal{E})$  iff  $e \in \mathcal{E}$ . If  $a \in e$  then  $e \in \mu_a(\mathcal{E})$  iff  $e \notin \mathcal{E}$ . So,  $\mu_a$  is an operation  $\mathcal{G}(X) \rightarrow \mathcal{G}(X)$ ; it is called the local complementation in the point  $a$ . It is seen that

$$\mu_a(G) = G \div K_{\{a\}, X \setminus \{a\}}. \quad (11)$$

Consequently, with the help of (7), we arrive to the definition introduced in [4]:

**PROPOSITION 1.2.** *Let  $G_1, G_2 \in \mathcal{G}(X)$ .  $G_1 \approx G_2$  iff there is a sequence  $a_1, \dots, a_k$  of elements of  $X$  such that  $G_2 = \mu_{a_k}(\dots \mu_{a_1}(G_1) \dots)$ .*

Clearly,  $\cong$  is also an equivalence relation on  $\mathcal{G}(X)$ . But  $\approx$  and  $\cong$  are essentially distinct:

- (i) Let  $a \in X$ ,  $|X| \geq 2$ ; then  $N_X \approx K_{\{a\}, X \setminus \{a\}}$ ,  $N_X \not\approx K_{\{a\}, X \setminus \{a\}}$ .
- (ii) Let  $A, B \subsetneq X$ ,  $|A| = |B| > 1$ . Then  $G_1 = K_A \cup N_{X \setminus A} \cong K_B \cup N_{X \setminus B} = G_2$ . It is seen that if  $A \cup B \neq X$  then  $G_1 \div G_2 \notin \mathcal{D}(X)$  and then  $G_1 \not\approx G_2$ . Take, in particular,  $A = \{1, 2\}$ ,  $B = \{2, 3\}$ ,  $X = A \cup B \cup \{4\}$ . Then  $G_1 \div G_2$  is the  $L_3$ -path  $1 - 2 - 3$  not in  $\mathcal{D}(X)$ .

The following is easy to prove

**PROPOSITION 1.3.** *Let  $G_1, G_2 \in \mathcal{G}(X)$ .*

- (i) *If  $G_1 \cong G_2$  then  $\varkappa(G_1) \cong \varkappa(G_2)$ .*
- (ii) *If  $G_1 \approx G_2$  then  $\varkappa(G_1) \approx \varkappa(G_2)$ .*

PROOF. (i) is evident: if a bijection  $f: X \rightarrow X$  maps  $\mathcal{E}_1$  onto  $\mathcal{E}_2$ , where  $G_i = \langle X, \mathcal{E}_i \rangle$  for  $i = 1, 2$ , then  $f$  maps  $\wp_2(X) \setminus \mathcal{E}_1$  onto  $\wp_2(X) \setminus \mathcal{E}_2$  as well.

Ad (ii): Note that for any  $G \in \mathcal{G}(X)$  we have  $\varkappa(G) = K_X \div G$ . So, suppose  $G_1 \approx G_2$  i.e.  $G_2 = G_1 \div H$  with  $H \in \mathcal{D}(X)$ . Then  $\varkappa(G_2) = G_2 \div K_X = (G_1 \div H) \div K_X = (G_1 \div K_X) \div H = \varkappa(G_1) \div H$ . Finally,  $\varkappa(G_1) \approx \varkappa(G_2)$ .  $\square$

Clearly,  $\cong$  is a congruence with respect to the operation  $\cup$ : if  $G_1 \cong G'_1$  and  $G_2 \cong G'_2$  (with suitably disjoint sets of vertices) then  $G_1 \cup G_2 \cong G'_1 \cup G'_2$ ; it is evident. Unhappily,  $\approx$  is not a congruence. Indeed, it is known that  $K_2 \cup K_1 \approx K_3$  and  $K_3 \cup K_1 \approx K_4$ . But  $(K_2 \cup K_1) \cup K_1 = K_2 \cup N_2 \approx L_4$  and it is known also that  $K_4 \not\approx L_4$ .

**LEMMA 1.4.** *Let  $G_1, G_2, G_3 \in \mathcal{G}(X)$  and  $G_1 \approx G_3 \cong G_2$ . Then there is  $G'_3 \in \mathcal{G}(X)$  such that  $G_1 \cong G'_3 \approx G_2$ .*

PROOF. Let  $f$  be a bijection that maps  $G_3$  onto  $G_2$ , let  $G_1 = G_3 \div H$  with  $H \in \mathcal{D}(X)$ . Then  $f(H) \in \mathcal{D}(X)$  and  $G_2 \div f(H) = f(G_3) \div f(H) = f(G_3 \div H) = f(G_1)$ . With  $G'_3 = G_2 + f(H)$  we get our claim.  $\square$

As a consequence of the above and of 1.1 we get that the relation  $\sim$  defined on  $\mathcal{G}(X)$  by the formula

$$G_1 \sim G_2: \iff G_1 \approx G_3 \cong G_2 \text{ for a graph } G_3 \quad (12)$$

is an equivalence relation. This is our main subject of the paper. For small values of  $n = |X|$  the classification of the elements of  $\mathcal{G}(X)$  with respect to the relation  $\sim$  is known (cf. [4]). Clearly,  $K_1 = N_1$  and  $K_2 = L_2 \approx N_2$ .

$n = 2 : \mathcal{G}(X)/\sim = \{K_2, N_2\}$ .

$n = 3 : \mathcal{G}(X)/\sim = \{[K_3]_\sim, [N_3]_\sim\}$  (cf. (9)).

$n = 4 : \mathcal{G}(X)/\sim = \{[K_4]_\sim, [N_4]_\sim, [L_4]_\sim\}$ .

The elements in the classes enumerated above are pairwise distinct. Moreover,  $L_4 \sim L_2 \cup N_2 \sim L_3 \cup N_1$ . The case  $n = 5$ , which is also already solved, will be quoted in the following.

Directly from (12) and 1.3 we have

**PROPOSITION 1.5.** *Let  $G_1, G_2 \in \mathcal{G}(X)$ . If  $X_1 \sim X_2$  then  $\kappa(G_1) \sim \kappa(G_2)$ .*

For a graph  $G = \langle X, \mathcal{E} \rangle$  and  $Z \subset X$  we write  $G \upharpoonright Z = \langle Z, \mathcal{E} \cap \wp_2(Z) \rangle$ . As a convenient tool for determining which graphs are *not*  $\sim$ -related we give

**LEMMA 1.6.** *If  $G_1, G_2 \in \mathcal{G}(X)$ ,  $Z \subset X$ , and  $G_1 \approx G_2$  then  $G_1 \upharpoonright Z \approx G_2 \upharpoonright Z$ .*

PROOF. Let  $\mathcal{Z}_i = \mathcal{E}_i \cap \wp_2(Z)$  for  $i = 1, 2$ . Assume that  $G_1 \approx G_2$ . So, there is a sequence of local complementations  $\mu_{a_1}, \dots, \mu_{a_k}$  which composition maps  $\mathcal{E}_1$  onto  $\mathcal{E}_2$ . Note that if  $a_j \notin Z$  then  $\mu_{a_j}(\mathcal{E} \cap \wp_2(Z)) = \mathcal{E} \cap \wp_2(Z)$  and if  $a_j \in Z$  then local complementation  $\mu_{a_j}$  maps  $\mathcal{E} \cap \wp_2(Z)$  onto  $(\mu_{a_j}(\mathcal{E})) \cap \wp_2(Z)$ , for every  $\mathcal{E} \subset \wp_2(X)$ . Consequently, a sequence of local complementations in points of  $Z$  maps  $\mathcal{Z}_1$  onto  $\mathcal{Z}_2$  and we are done by 1.2.  $\square$

And afterwards, as important invariants we obtain

**PROPOSITION 1.7.** *Let  $G_0$  be a graph on  $k$  vertices,  $k < |X|$ . For  $G \in \mathcal{G}(X)$  we write*

$$G/G_0 = \{Z \in \wp_k(X) : G \upharpoonright Z \sim G_0\} \quad (13)$$

*Let  $G_1, G_2 \in \mathcal{G}(X)$ . If  $G_1 \sim G_2$  then there is a bijection  $f$  of  $X$  which maps the family  $G_1/G_0$  onto  $G_2/G_0$ .*

**PROPOSITION 1.8.** *Let  $G_0 \in \mathcal{G}(Z_0)$ ,  $|Z_0| = k < |X|$ . For  $G \in \mathcal{G}(X)$  we define*

$$\#(G; G_0) := |G/G_0|. \quad (14)$$

- (i) *Let  $G_1, G_2 \in \mathcal{G}(X)$ . If  $G_1 \sim G_2$  then  $\#(G_1; G_0) = \#(G_2; G_0)$ .*
- (ii)  *$\#(\mathcal{K}(G); \mathcal{K}(G_0)) = \#(G; G_0)$ .*

Proof is straightforward and is omitted here, but the facts formulated in 1.7 and 1.8 will be frequently used.

The following notation will be also convenient

$$G_0^n = G_0 \cup N_{n-k} \text{ where } k = |X_0|, G_0 \in \mathcal{G}(X_0).$$

Analogous notation will be used when only the  $\cong$ -type of  $G_0$  will be important, e.g.  $L_k^n, (K_2 \cup K_2)^6$  etc.

From (9) we can relatively easily compute the following formulas

	assume that	$n > 3$	
$\#(L_n; K_3) = (n-2)(n-3) \quad (15)$			$\#(C_n; K_3) = n(n-4) \quad (20)$
$\#(L_n; K_4) = \binom{n-3}{2} \quad (16)$			$\#(C_n; K_4) = \frac{n(n-5)}{2} \quad (21)$
$\#(L_n; K_m) = 0 \text{ for } m > 4 \quad (17)$			$\#(C_n; K_m) = 0 \text{ for } m > 4 \quad (22)$
$\#(L_n; N_3) = (n-2) + \binom{n-2}{3} \quad (18)$			$\#(C_n; N_3) = n + \frac{n(n-4)(n-5)}{6} \quad (23)$
$\#(L_n; N_m) = \binom{n-m+1}{m} \quad (19)$			$\#(C_n; N_m) = \frac{n \binom{n-m-1}{m-1}}{m} \quad (24)$

PROOF. The reasoning is standard, we shall only show in several examples how to handle with the formulas like these.

Ad (15) It is impossible to find  $K_3$  in  $L_n$ . So, we search for  $K_2 \cup K_1$  within  $L_n$ . These are obtained by an edge  $e$  of  $L_n$  and a vertex  $v$  sufficiently far from the endpoints of  $e$ . If  $e$  is the first or the last in the path  $L_n$  then  $v$  can be chosen in  $n-3$  ways; if  $e$  is intermediate ( $L_n$  contains  $n-3$  such edges) then  $v$  can be chosen in  $n-4$  ways.

Ad (16) We must determine all the  $K_2 \cup K_2$  subgraphs in  $L_n$ . So, we must find a pair of edges  $\{i_1, i_1+1\}, \{i_2, i_2+1\}$  such that  $1 \leq i_1, i_1+3 \leq i_2 \leq n-1$ . Elementary combinatorics justifies that the number of such pairs  $(i_1, i_2)$  is as claimed.

Ad (17) If a graph contains  $K_A \cup K_B$  with  $|A \cup B| \geq 5$  then it contains a point of rank at least 3; clearly,  $L_n$  has no such a point.

Ad (18) We must determine all bipartite  $K_{\{i_1\}, \{i_2, i_3\}}$  and all  $N_{i_1, i_2, i_3}$  contained in  $L_n$ . In the first case we choose  $1 < i_1 < n$  (in  $n-2$  ways) and set  $i_2 = i_1 - 1, i_3 = i_1 + 1$ . In the second case we look for sequences  $1 \leq i_1, i_1 + 2 \leq i_2, i_2 + 2 \leq i_3 \leq n$ .  $\square$

From this, by analogous reasonings, we get more complex formulas, e.g.

$$\begin{aligned} \#(L_{n_1} \cup \dots \cup L_{n_k} \cup C_{l_1} \cup \dots \cup C_{l_t}; K_3) = \\ \sum_{i=1}^k [(n_i - 2)(n_i - 3) + (n_i - 1)(n - n_i)] + \sum_{i=1}^t [l_i(l_i - 4) + l_i(n - l_i)], \end{aligned} \quad (25)$$

$$\begin{aligned}
\#(L_{n_1} \cup \dots \cup L_{n_k} \cup C_{l_1} \cup \dots \cup C_{l_t})^n; K_4) = \\
\sum_{i=1}^k \binom{n_i-3}{2} + \sum_{j=1}^t \frac{l_j(l_j-5)}{2} + \\
\sum_{1 \leq i_1 < i_2 \leq k} [(n_{i_1-1})(n_{i_2} - 1) + \sum_{1 \leq j_1 < j_2 \leq t} l_{j_1} l_{j_2} + \sum_{i=1}^k \sum_{j=1}^t (n_i - 1) l_j]; \quad (26)
\end{aligned}$$

we assume here that  $l_1, \dots, l_t > 3$ .

## 2 Inductive enumerating of $\sim$ -classes

We begin with the following ‘inductive’ observation. Let  $X_0$  be a set,  $w \notin X_0$ ,  $X = X_0 \cup \{w\}$ . Then each  $G \in \mathcal{G}(X)$  can be presented in the form

$$\begin{aligned}
G = G_0 \div K_{w,Z}, \text{ where } Z \subset X_0, G_0 \in \mathcal{G}(X_0), \\
Z = \{x \in X_0 : \{w, x\} \in G\} \text{ and } G_0 = G \upharpoonright X_0. \quad (27)
\end{aligned}$$

Assume that  $G$  has form (27).

**LEMMA 2.1.** *If  $G_0 \approx G'_0$ ,  $G_0 = G'_0 \div K_{A, X_0 \setminus A}$ , then  $G \approx G'_0 \div K_{w, A \div Z}$ .*

PROOF. It suffices to note that  $K_{A, X \setminus A} \div K_{A, X_0 \setminus A} = K_{w, A}$  and  $K_{w, A} \div K_{w, Z} = K_{w, A \div Z}$ . Then  $G = G_0 \div K_{w, Z} = G'_0 \div K_{A, X_0 \setminus A} \div K_{w, Z} = G'_0 \div K_{A, X_0 \setminus A} \div K_{A, X \setminus A} \div K_{A, X \setminus A} \div K_{w, Z} = G'_0 \div K_{w, A} \div K_{w, Z} \div K_{A, X \setminus A} = (G'_0 \div K_{w, A \div Z}) \div K_{A, X \setminus A}$ .  $\square$

In consequence, to determine all the types of graphs on  $X$  it suffices to choose a point  $w \in X$ , and for each type  $G_0$  of a graph on  $X_0 = X \setminus \{w\}$  enumerate all, up to an isomorphism of  $G_0$ ,  $k$ -subsets  $Z$  of  $X_0$  such that  $2k \leq |X|$ . Each type of a graph on  $X$  is realized as  $G_0 \div K_{w, Z}$  with so obtained  $G_0$ ’s and  $Z$ ’s. To complete the task it suffices to verify which of the graphs on the list composed so far are  $\sim$ -equivalent and which are not. Let us illustrate how this procedure works and let us apply it to the case  $n = |X| = 6$ .

Let us quote the following

**PROPOSITION 2.2** ([4, page 204]). *There are exactly 7  $\sim$ -types of graphs on 5 vertices. These are the following:*

$$\begin{aligned}
5:1 \ K_5, \approx K_4^5; \quad 5:2 \ N_5 = \varkappa(K_5); \quad 5:3 \ C_5; \quad 5:4 \ L_2^5 = K_2^5; \\
5:5 \ \varkappa(L_2^5), \approx K_3^5; \quad 5:6 \ L_3^5; \quad 5:7 \ \varkappa(L_3^5) \approx L_5.
\end{aligned}$$

Our goal (one of some) is to prove the following

**THEOREM 2.3** (6-graphs). *Let  $G$  be a graph on 6 vertices. Then  $G$  is  $\sim$ -equivalent to one of the following graphs.*

$$\begin{aligned}
6:1 \ K_6; \quad 6:2 \ N_6; \quad 6:3 \ L_2^6; \quad 6:4 \ \varkappa(L_2^6), \approx K_4^6, C_3 \cup L_3; \\
6:5 \ (L_2 \cup L_2)^6; \quad 6:6 \ C_5^6; \quad 6:7 \ L_3^6; \quad 6:8 \ K_3^6; \\
6:9 \ (L_3 \cup L_2)^6; \quad 6:10 \ L_4^6; \quad 6:11 \ L_3 \cup L_3; \quad 6:12 \ \varkappa(L_3^6); \approx (C_3 \cup L_2)^6 \\
6:13 \ C_4^6, \approx \varkappa(C_6); \quad 6:14 \ L_5^6; \quad 6:15 \ L_6; \quad 6:16 \ C_6, \approx \varkappa(L_2 \cup L_2 \cup L_2).
\end{aligned}$$

*No two graphs in this list are  $\sim$ -equivalent.*

PROOF. Let us assume that graphs classified here are defined on the set  $\{1, 2, 3, 4, 5, 6\}$ , write  $w = 6$  and apply 2.1. Let  $X_0 = \{1, 2, 3, 4, 5\}$ . So, we obtain the list of graphs of the form  $G_0 \div K_{w,Z}$ ,  $G_0$  is one from among those enumerated in 2.2 and  $Z \in \wp_1(X_0) \cup \wp_2(X_0) \cup \{\emptyset\}$ . In what follows we shall indicate mainly sets of edges of corresponding graphs.

To shorten notation we shall also write " $i.j = \varkappa(i'.j')$ ", if  $G = \varkappa(G')$ ,  $G$  stands on the position  $i.j$  in the list below, and  $G'$  has the position  $i'.j'$ . Analogous meaning has notation " $i.j = i'.j'$ ", " $G = i.j$ ", " $i.j \sim i'.j'$ ", and " $G \sim i.j$ ". The symbol  $\bigcirc$  indicates the case when the resulting graph already belongs to those enumerated through 6:1-6:16 or it coincides with a graph considered earlier. So, it means 'there is nothing to prove in this case'.

1. Let  $G_0$  in 5:1

- 1.0  $Z = \emptyset$ . Then  $G = \wp_2(X_0) = K_{X_0} \cup K_{\{6\}} \sim K_X$ , so  $G$  has the type  $K_6$ , declared in 6:1.  $\bigcirc$
- 1.1  $Z = \{5\}$ . Then  $G = \wp_2(X_0) \cup \{\{5, 6\}\}$  and  $G \sim \varkappa(L_2^6)$ , declared in 6:4.  $\bigcirc$   
Moreover,  $G \sim C_3 \cup L_3$ .
- 1.2  $Z = \{4, 5\}$ . Then  $G = \wp_2(X_0) \cup \{\{5, 6\}, \{4, 6\}\}$  and  $G \sim 5.1, 7.5$ .

2. Let  $G_0$  in 5:2

- 2.0  $Z = \emptyset$ . Then  $G = \emptyset$ , so  $G$  has the type  $N_6$ , declared in 6:2.  $\bigcirc$
- 2.1  $Z = \{5\}$ . Then  $G = \{\{5, 6\}\}$ , so  $G$  has the type  $L_2^6$ , declared in 6:3.  $\bigcirc$
- 2.2  $Z = \{4, 5\}$ . Then  $G = \{\{5, 6\}, \{4, 6\}\}$ , so  $G$  has the type  $L_3^6$ , declared in 6:7.  $\bigcirc$

3. Let  $G_0$  in 5:3. Say,  $G$  is the cycle  $1 - 2 - 3 - 4 - 5 - 1$ . Considering the automorphism group of  $G_0$  we see that it suffices to consider the following cases only.

- 3.0  $Z = \emptyset$ . Then  $G$  has the type  $C_5^6$ , declared in 6:6.  $\bigcirc$
- 3.1  $Z = \{1\}$ . Then  $G \sim 6.5$
- 3.2  $Z = \{1, 2\}$ . Then  $G \sim 7.1$ .
- 3.3  $Z = \{1, 3\}$ . Then  $G \sim 4.5$

4. Let  $G_0$  in 5:4. Say, the unique edge of  $G_0$  is  $\{1, 2\}$ . It is seen that it suffices to consider the following sets  $Z$ .

- 4.0  $Z = \emptyset$ . Then  $G$  has the type  $L_2^6$ , equal to 2.1.  $\bigcirc$
- 4.1  $Z = \{1\}$ . Then  $G$  is, in fact, the path  $6 - 1 - 2$ , so it has the type  $L_3^6$  (and  $G = 2.2$ ).  $\bigcirc$
- 4.2  $Z = \{3\}$ . Then  $G$  has two, disjoint, edges: it is  $(L_2 \cup L_2)^6$ , declared in 6:5.  $\bigcirc$
- 4.3  $Z = \{1, 2\}$ . Then  $G$  is the triangle  $1, 2, 6$ , so it is  $K_3^6$ , declared in 6:8.  $\bigcirc$
- 4.4  $Z = \{3, 4\}$ . Then  $G$  consists of the 3-path  $3 - 6 - 4$  and the edge  $1 - 2$  and thus it is  $(L_3 \cup L_2)^6$ , declared in 6:9.  $\bigcirc$
- 4.5  $Z = \{1, 3\}$ . Then  $G$  consists of the 4-path  $2 - 1 - 6 - 3$ ; it is  $L_4^6$ , declared in 6:10.  $\bigcirc$



5. Let  $G_0$  in 5:5. Then  $G_0 = \varkappa(G'_0)$ , where  $G'_0$  is given in 5:4. Consequently,  $G \sim \varkappa(G')$ , where  $G'$  is a one among those enumerated through 4.0–4.5.

5.0  $G = \varkappa(L_2^6)$ .  $G = 1.1$ . ○

5.1  $G = \varkappa(L_3^6)$ , declared in 6:12. ○  
Moreover,  $G \sim (C_3 \cup L_2)^6$ .

5.2  $G = \varkappa((L_2 \cup L_2)^6) \sim 6.4$ .

5.3  $G = \varkappa(K_3^6) \sim 4.3$ .

5.4  $G = \varkappa((L_3 \cup L_2)^6) \sim 4.4$ .

5.5  $G = \varkappa(L_4^6) \sim 7.1$ ; moreover  $G \sim L_2 \cup L_4$ .

6. Let  $G_0$  in 5:6. One can assume that  $G_0$  is the path  $1-2-3$ , and  $4, 5$  are isolated. The following cases must be considered.

6.0  $Z = \emptyset$ . Then  $G = L_3^6$  (as in 4.1 and 2.2). ○

6.1  $Z = \{4\}$ . Then  $G$  consist of the 3-path  $1-2-3$  and the edge  $4-6$ , so  $G = (L_2 \cup L_3)^6$ , as in 4.4. ○

6.2  $Z = \{1\}$ . Then  $G$  is the path  $6-1-2-3$ , so  $G = L_4^6 = 4.5$ . ○

6.3  $Z = \{2\}$ ; then  $G \approx 4.1$ .

6.4  $Z = \{4, 5\}$ . Then  $G$  consist of two disjoint 3-paths  $1-2-3$  and  $4-6-5$ , so  $G = L_3 \cup L_3$ , declared in 6:11. ○

6.5  $Z = \{4, 1\}$ . Then  $G$  is the path  $4-6-1-2-3$ , so  $G = L_5^6$ , declared in 6:14. ○

6.6  $Z = \{4, 2\}$ ; then  $G \sim 4.5$ .

6.7  $Z = \{1, 3\}$ . Then  $G$  is the closed cycle  $5-1-2-3-5$ , so  $G = C_4^6$ , declared in 6:13. ○

6.8  $Z = \{1, 2\}$ . Then  $G \sim 4.4$ .

7. Let  $G_0$  in 5:7. One can assume that  $G_0$  is the 5-path  $1-2-3-4-5$ . The following sets  $Z$  must be considered.

7.0  $Z = \emptyset$ . Then  $G = L_5^6 = 6.5$ . ○

7.1  $Z = \{1\}$ . Then  $G$  is the path  $6-1-2-3-4-5$ , so  $G = L_6$ , declared in 6:15. ○

7.2  $Z = \{2\}$ . Then  $G \sim 6.1$  and  $G \sim 7.6$ .

7.3  $Z = \{3\}$ . Then  $G \sim 3.1$ .

7.4  $Z = \{1, 5\}$ . Then  $G$  is the closed cycle  $6-1-2-3-4-5-6$ , i.e.  $G = C_6$ , declared in 6:16. ○

7.5  $Z = \{1, 2\}$ : cf. 1.2. ○

7.6  $Z = \{1, 3\}$ : cf. 7.2. ○

7.7  $Z = \{1, 4\}$ : Then  $G = 3.1$ .

7.8  $Z = \{2, 3\}$ . Then  $G \sim 7.1$ .

7.9  $Z = \{2, 4\}$ : Then  $G \sim 4.2$ .

Ad 1.1 Note that  $G \div K_{\{1,5\},\{2,3,4,6\}} \cong C_3 \cup L_3$ . Moreover,  $G \div K_{\{6\},\{1,2,3,4,5\}} \cong \varkappa(L_2^6)$ .

Ad 1.2  $G \div K_{\{1,5\},\{2,3,4,6\}}$  is the path  $5-1-6-4$  connected with the triangle  $4, 3, 2$  i.e.  $G \sim 7.5$ . Next,  $G \div K_{\{6\},\{1,2,3,4,5\}} = \varkappa(L_3^6) = \varkappa(4.1) = 5.1$ .

Ad 3.1  $G \div K_{\{1,4\},\{2,3,5,6\}}$  is the path  $1-3-2-4-6$  i.e.  $L_5^6 = 6.5$ .

Ad 3.2  $G \div K_{\{1,3\},\{2,4,5,6\}}$  is the path  $1-4-5-3-6-2$  i.e.  $= L_6 = 7.1$ .

$G$	6:1	6:2	6:3	6:4	6:6	6:5	6:7	6:8	6:9	6:10	
$\#(G; K_3)$	20	0	4	16	10	8	6	10	10	8	
	$G$					6:12	6:11	6:13	6:14	6:15	6:16
	$\#(G; K_3)$					14	12	8	10	12	12

  

$G$	6:6	6:8	6:9	6:14	6:5	6:10	6:13	6:11	6:15	6:16
$\#(G; K_4)$	0	3	2	1	1	0	0	4	3	3
$\#(G; N_4)$	0	3	2	1	4	3	3	1	0	0

Table 1: Parameters  $\#(G; K_3)$ ,  $\#(G; K_4)$ , and  $\#(G; N_4)$  of the graphs  $G$  defined in 2.3

Ad 3.3  $G \div K_{\{1,3\},\{2,4,5,6\}}$  is the  $L_4$ -path  $1 - 4 - 5 - 3 = 4.5$ .

Ad 5.1  $G \div K_{\{3,4,5\},\{1,2,6\}}$  is the union of one edge  $2 - 6$  and the triangle  $3, 4, 5$ , so it is  $(L_2 \cup C_3)^6$ .

Ad 5.2  $G \div K_{\{1,2,4\},\{3,5,6\}}$  is the union of two  $L_3$ -paths:  $1 - 4 - 2$  and  $3 - 5 - 6 = 6.4$ .

Ad 5.3  $G \div K_{\{1,2,6\},\{3,4,5\}}$  is the triangle  $3, 4, 5$ , so we obtain  $K_3^6 = 4.3$ .

Ad 5.4  $G \div K_{\{1,2,5\},\{3,4,6\}}$  is the union of the edge  $3 - 4$  and the  $L_3$ -path  $1 - 5 - 2$ .

Ad 5.5  $G \div K_{\{1,2,4\},\{3,5,6\}}$  is the path  $2 - 4 - 1 - 6 - 5 - 3 = 7.1$ . Moreover,  $G \div K_{\{4,5\},\{1,2,3,6\}}$  is the union of the  $L_4$ -path  $1 - 3 - 2 - 6$  and the edge  $4 - 5$ , so  $G \sim L_2 \cup L_4$ .

Ad 6.3  $G \div K_{\{2\},\{1,3,4,5,6\}}$  is the  $L_3$ -path  $4 - 2 - 5$ .

Ad 6.6  $G \div K_{\{2\},\{1,3,4,5,6\}}$  is the  $L_4$ -path  $5 - 2 - 4 - 6$ .

Ad 6.8  $G \div K_{\{2\},\{1,3,4,5,6\}}$  is the union of the  $L_3$ -path  $4 - 2 - 5$  and the edge  $1 - 6$ .

Ad 7.2  $G \div K_{\{2,4\},\{1,3,5,6\}}$  is the union of the  $L_3$ -path  $1 - 4 - 6$  and the edge  $2 - 5$ , i.e.  $= 4.4$ . Next,  $G \div K_{\{3\},\{1,2,4,5,6\}} \cong 7.6$ .

Ad 7.3  $G \div K_{\{1,4,6\},\{2,3,5\}}$  is the 5-cycle  $1 - 3 - 2 - 6 - 5 - 1$  with the edge  $2 - 4$  added, so it is  $\cong 3.1$ .

Ad 7.8  $G \div K_{\{1,4,6\},\{2,3,5\}}$  is the  $L_6$ -path  $4 - 2 - 3 - 1 - 5 - 6$ .

Ad 7.9  $G \div K_{\{2,4\},\{1,3,5,6\}}$  is the union of two disjoint edges  $1 - 4$  and  $2 - 5$ , which is  $= 4.2$ .

To complete the proof we analyse Table 1 (note that  $\#(G; N_3) = 20 - \#(G; K_3)$ , as  $G$  contains 20 subgraphs on 3 vertices). From 1.8 we see that only two cases must be distinguished by other methods. Here we apply 1.7. To distinguish 6:10 and 6:13 we note that  $L_4^6/N_4$  consists of three 4-sets with a common 2-set; while no such a common subset exists for the elements of  $C_4^6/N_4$ . Similarly, to distinguish 6:15 and 6:16 we note that  $L_6/K_4$  consists, analogously, of three subsets with the common 2-set, and this is not true for  $C_6/K_4$ .  $\square$

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